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Abstract

We present a method for blind recovery of network made out of a tree of 1D homogeneous waveguides with the same physical characteristics using reflectogram and transferogram(s).

Keywords: Inverse problem, Topology recovering, Quantum graph, Reflectometry, Wire analysis

1 Introduction

We consider an unknow quantum graph $\mathcal{G}$ (see [2]) equipped with a wave operator along its branches and some transmission conditions on its nodes connecting together the quantities evaluated on the branches. Our graph is a rooted tree graph, where all branches are oriented from a root-point Inp to end-points Out_k ($k = 1...K$). We will consider a maximum of two consecutive nodes between Inp and any Out_k and at least a node in $\mathcal{G}$.

We will now explain how a wave V propagates along the graph $\mathcal{G}$:

- On each branch of $\mathcal{G}$ the wave satisfiy an homogeneous wave's equation
$$\partial^2_{tt} V - c^2 \partial^2_{xx} V = 0,$$
where t denotes the time and x the abscissa along the considered branch. The celerity c of the waves is supposed to be a known constant.

- Following Kirchhoff's rules, V is continuous on $\mathcal{G}$ and at each node J
$$\partial_x V|_{\mathbf{e}_{j_0}}(J) = \sum_{j_k=j_1}^{j_K} \partial_x V|_{\mathbf{e_{j_k}}}(J),$$
where $\mathbf{e_{j_k}}$ are the branches connected to J, with $\mathbf{e}_{j_0}$ the branch closest to Inp.

- On Inp, we have an impedance boundary condition
$$\partial_t V(\mathrm{Inp}, t) - c\frac{Z_u}{Z_c}\partial_x V(\mathrm{Inp}, t) = (\partial_t u)(t)$$
where the constant Z_u and $u \in H^1_{\mathrm{loc}}(\mathbb{R})$ are known. The unknown characteristic impedance Z_c is supposed to be constant.

- At each Out_k we have an impedance condition
$$\partial_t V + c\frac{Z_k}{Z_c}\partial_x V = 0,$$
where Z_k is an unknown constant.

2 Graph recovery problem

Reflectometry and transferometry methods can be applied to any practical electrical or acoustic network. The reflectogram is the following Steklov operator :

$$u(t) \mapsto \mathfrak{R}(t) := V(\mathrm{Inp}, t),$$

whereas transferograms are operators for $k = 1...K$:

$$u(t) \mapsto \mathfrak{T}_k(t) := V(\mathrm{Out}_k, t).$$

We suppose that we can control u. With a known celerity c and the input load Z_u, the reflectogram and optionally some transferograms, we want to recover $\mathcal{G}$ that is to say to determine

- the number of nodes and end-points, and their ordering (topology),
- the lenght ℓ_j of all branches,
- the end-points load Z_k,
- the characteristic impedance Z_c.

3 Injectivity

There can exists several quantum graph with the same reflectogram, so we will make two hypothesis. Firstly, no scatterer (node or end-points) have the same distance from Inp, to ensure they can be dissociated. Secondly, no Z_k is equal to Z_c to ensure waves are reflected on the end-point. We will suppose that $Z_k > Z_c$ is always statisfied.

4 Scattering

We can choose the excitation signal u such it is a peak function (sole local maximum). It propagates at celerity c along a branch until it meets a scatterer. $\mathfrak{R}$ is the sum of attenuated u-shaped peaks, i-e of the form $\sum_p S_p u(t - t_p)$ where where each (t_p, S_p) - called echo - corresponds to the duration and the amplitude attenuation of a propagation of u in $\mathcal{G}$ through transmissions T and reflections Γ looping on Inp. T_k is similarly generated, with propagations from Inp to Out_k. An echo amplitude S_p gives the nature of its contained scatterers (order for a node, load for an end-point), its abscissa the path length between the observation point (Inp for $\mathfrak{R}$ and Out_k for $\mathfrak{T}_k$) and the scatterer.

The algorithm presented in [1] identifies echoes in a complex reflectogram and associates them with unknown scatterers in $\mathcal{G}$, giving their nature and location. It runs iteratively, dispelling ambiguities from peaks overlaping and accumulated reflections.

But this method requires knowledge of Z_c and supposes that $Z_c = Z_u$ (no reflections at Inp). It can be enhanced by the use of transferograms.

5 Algorithm

5.1 Recovering Z_c

We simply recover Z_c from the reflectogram at origin where we see an echo (called mismatch echo) of amplitude $T_u = (1 - \Gamma_u)$ with $\Gamma_u = (Z_u - Z_c)(Z_u + Z_c)$. Of course if $Z_c = Z_u$ then the mismatch peak is null.

5.2 Recovering the first node

The first echo observe in $\mathfrak{R}$ after the mismatch echo have for abscissa $2\ell_0/c$ and for amplitude $T_u(1+\Gamma_u)\Gamma_0$ with $\Gamma_0 = (2/m_0 - 1)$ where m_0 is the order of the first node J_0. We thus recover ℓ_0 and m_0.

5.3 Using the transferograms

If the amplitude of the first echo (t_1, S_1) of $\mathfrak{T}_k$ is above $4T_u(\Gamma_0 + 1)/3$, then Out_k is directly connected to J_0. Thus we have $S_1 = T_u(2/m_0)(1 + \Gamma_k)$ with $\Gamma_k = (Z_k - Z_c)(Z_k + Z_c)$, so we recover Z_k. The length of the J_0 to Out_k branch is $(ct_1 - \ell_0)$. If S_1 is under $T_u(\Gamma_0+1)$, a node exists between J_0 and Out_k. This recovered topology can remove branch location ambiguities for an unknown scatterer with the reflectogram.

5.4 Using the reflectogram

We use the algorithm developed in [1] to continue the analysis of $\mathfrak{R}$. We changed the procedure to use informations from the transferograms, and adapt to $Z_u \neq Z_c$. Indeed, we need to apply a $T_u(1 - \Gamma_u)$ factor to all $\mathfrak{R}$ echoes amplitude and to consider reflexions on Inp when discriminating between echoes.

This method achieves an error-free topology reconstruction if some technical hypothesis on u are fullfiled. ℓ_j are retrieved with an accuracy decreasing when farther from Inp (relative error under 5% from 350 simulations), as are Z_c (under 0.1%) and Z_k (under 10%). Better determination is possible by optimizing the all lengths $\widetilde{\boldsymbol{\ell}}$ and all loads $\widetilde{\mathbf{Z}}$ vectors such that they minimize the functional

$$J(\boldsymbol{\ell}, \mathbf{Z}) = \int_0^{8\ell_{max}/c} \frac{|\mathfrak{R}(t) - \mathfrak{R}_{\boldsymbol{\ell},\mathbf{Z}}(t)|^2}{3\ell_{\max}/c + t^2} dt$$

where ℓ_{max} is the maximum ℓ_j from previous steps and $\mathfrak{R}_{\boldsymbol{\ell},\mathbf{Z}}$ the simulated reflectogram using the recovered topology with $\boldsymbol{\ell}$ and $\mathbf{Z}$. We look for the minimum of J with Newton's algorithm, initializing $\widetilde{\boldsymbol{\ell}}$ and $\widetilde{\mathbf{Z}}$ with the previously recovered values.

6 Applications

This algorithm can be used for recovering unknown electrical networks, with one reflectometry device and optionnal transferometry transcievers on end-points. Removing the condition on Z_u makes the algorithm more resilient to real-life implementation limitations.

References

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